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An improved theory of gravitation: II

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Abstract. The theory of gravitation of a previous paper is presented in a deductive and more rigorous form. The assumptions made about the space–time metric, the scalar gravitational potential and the special (Newtonian) charts are summarized. An action principle is stated, and the conservation laws of energy–momentum and angular momentum are derived. Lagrangian densities for the gravitational field are found by assuming that weak gravitational waves propagate at the speed of light. The assumption that gravitational energy is not itself a source of the gravitational field leads, as in a previous paper, to a theory that is at present observationally indistinguishable from Einstein's; the opposite assumption leads to a distinguishable theory. The interactions of the gravitational field with the electromagnetic field and with an ideal fluid are discussed. The simplicity of the theory (space–time formally flat and one scalar potential to describe the gravitational field) is emphasized.

1. Introduction

In a previous paper (Rastall 1968, to be referred to as I), we tried to build a theory of gravitation on assumptions that differ as little as possible from those of special relativity and the Newtonian theory. What we now attempt is a more rigorous deductive account. The approach will be field theoretical, which will avoid the difficulties encountered in I in dealing with particles. We begin by summarizing the more important results of I, separating the wheat from the goats.

We assumed in I that space–time is Riemannian with a metric g, and we postulated the existence of *Newtonian charts*, in which the spatial diagonal components of the metric are equal, the non-diagonal components are zero, and all components are determined by a single real function Φ, the *gravitational potential*. It was shown that the Newtonian charts almost always determine a time direction at each point: more precisely, if (U, χ) and (U, χ') are Newtonian charts, and if $X_0(p)$ and $X_0'(p)$ are the tangent vectors to their time-like coordinate curves at the point $p \in \mathrm{U}$, then $X_0'(p) = k_p X_0(p)$ for some constant k_p. Assuming that the gravitational potential is arbitrary to the extent of an additive constant (i.e. only differences of potential are measurable), we proved that the components of the metric are exponential functions of the potential. We proved also that, given any neighbourhood U of a point p and an orthonormal tetrad $\omega_\mu(p)$ at p, there is almost always at most one Newtonian chart (U, χ) whose coordinate curves have the $\omega_\mu(p)$ as tangent vectors at p. (These results are invalid in a few cases where the potential is a very simple function.)

At any point p in the domain of a chart, the tangent vectors of the coordinate curves form a basis of the tangent space at p. We showed (see I, equation (10)) that special Newtonian charts always exist whose tangent vectors are orthonormal with respect to the metric g at any point where the potential has the value Φ_0. Such charts are called Φ_0 *charts*, and from now on our Newtonian charts will always be Φ_0 charts (although Φ_0 will not always be the same).

It simplifies calculations to introduce a new metric η, with respect to which the tangent vectors of a chart are orthonormal at every point. Introducing η is equivalent to making a Φ-dependent change in the units of length and time. The units corresponding to η are called *Newtonian* (or Φ_0) units, while those corresponding to g are called *natural* units. A length measured in Φ_0 units is called a Φ_0 length, etc.

Particle dynamics was discussed in I. The paths of test particles are assumed to be geodesics of the metric g. A Φ-dependent change in the unit of mass, similar to the previous changes in the units of length and time, makes the equation of motion of a particle formally identical with the special-relativistic equation. The new unit of mass is again called a Newtonian or Φ_0 unit, and in the obvious way one defines Φ_0 units of all quantities whose dimensions involve only mass, length and time. We usually distinguish natural quantities by a suffix E: an exception is the gravitational potential Φ, which is always measured in natural units.

The components of the metric g in a Φ_0 chart, as given in I, equation (10), depend on two constants. One of these is determined by requiring that a slowly moving test particle in a weak gravitational field should behave as in Newtonian theory; the other from the assumption that the potential at a distance r from a fixed body tends asymptotically to $\Phi_1 - l/r$ as $r \to \infty$, where Φ_1 and l are constants, and l (the Φ_0 *gravitational radius* of the body) is proportional to the body's Φ_0 energy.

All the results listed so far seem suitable for incorporation in a more formal theory of gravitation. It is when one considers the gravitational effect of a particle on itself that troubles arise (by *particle* we mean of course a *point particle*). The potential at the position of a particle may well be undefined, just as in Newtonian theory. We therefore assumed in I that the potential due to a particle has no direct effect on the particle itself, although it does have an indirect effect because it changes the gravitational radii of neighbouring particles. It is also possible that the energy of a particle's gravitational field may itself act as a source of potential, and thus influence the particle. The situation here is obviously complex and unclear (see also I, § 9). The difficulties seem to be quite fundamental, and their resolution would require a precise definition of what one means by a particle. We know that this is a delicate matter, even in classical electrodynamics (Rohrlich 1965). It is possible that particles are essentially quantal phenomena that have no place in a classical theory (Dirac 1951). In this paper we shall not talk about particles (other than test particles—and in § 6 we shall see how to eliminate even these). Instead we shall develop a pure field theory.

2. Φ_0 charts

We assume, as in I, that space–time is a C^∞, 4-dimensional, pseudo-Riemannian manifold of signature 2 (Hicks 1965, Helgason 1962). Given any point p_0 of space–time, we assume that there exists an open set U, real constants Φ_0 and c_E, and a chart (U, χ) that belongs to the C^∞ differentiable structure, such that $p_0 \in U$, and such that the components of the metric g in (U, χ) are given by

$$\left.\begin{aligned} g_{mn}(x) &= \delta_{mn} \exp\left[\frac{-2\{\Phi(x)-\Phi_0\}}{c_E^2}\right] \\ g_{\mu 0}(x) &= -\delta_{\mu 0} \exp\left[\frac{2\{\Phi(x)-\Phi_0\}}{c_E^2}\right] \end{aligned}\right\} \qquad (2.1)$$

for all $p \in U$, where $x = (x^0, x^1, x^2, x^3) = \chi(p)$ and $\Phi{:}\chi(U) \to R^1$ is smooth (that is C^∞). The constant c_E is the natural *speed of light*, Φ is the *gravitational potential* and (U, χ) is a Φ_0 *chart* (*on* U). We have shown (I, appendix) that in general the Φ_0 chart on U is determined up to a shift of origin and an orthogonal transformation of the spatial coordinates. That is, if (U, χ) and (U, χ') are Φ_0 charts and $\chi(p) = x$, $\chi'(p) = x'$, then $x'^k = b_{km}x^m + a^k$ for all $p \in U$, where the a^k and b_{km} are constants, and $b_{km}b_{kn} = \delta_{mn}$. Thus one has always the same freedom in choosing a Φ_0 chart as in choosing a Galilean chart in classical mechanics. Usually one has no more freedom than this: the most important exception is when $\Phi = \Phi_0$ (the Φ_0 charts become inertial charts and we are free to make Lorentz transformations).

It follows from (2.1) that if a Φ_0 chart on U exists, then a Φ_0' chart on U exists, for any constant Φ_0'. Since no physically significant statement can depend on an arbitrary choice of chart, we must take care that the predictions of the theory do not depend on the choice of a particular Φ_0 or, once we have chosen Φ_0, on a particular choice of Φ_0 chart.

If p is a space–time point, there exists a Φ_0 chart (U, χ) such that $p \in U$. Let $X_\mu(p)$ be the tangent vectors of this chart at p. Then a metric tensor $\eta(p)$ is defined at p by requiring that $\eta(p)(X_\mu(p), X_\nu(p)) = \eta_{\mu\nu}$, where $\eta_{mn} = \delta_{mn}$, $\eta_{\mu 0} = \eta_{0\mu} = -\delta_{\mu 0}$. It is easy to see that in general $\eta(p)$ is uniquely defined, for fixed Φ_0, independently of the choice of Φ_0 chart. Since the Φ_0 charts cover space–time, one can define a metric tensor field η globally (i.e. on the whole of space–time). We note that η depends on Φ_0.

The Φ_0 charts are related to Φ_0 lengths and times in the same way that inertial charts are related to natural lengths and times in special relativity. For example, if two points have Φ_0 coordinates x and x', and if $x^0 = x'^0$, then the Φ_0 length of the line segment joining them is $|\boldsymbol{x}-\boldsymbol{x}'| = \{(x'^k - x^k)(x'^k - x^k)\}^{1/2}$. Again, if $t = x^0/c_{\mathrm{E}}$ is the Φ_0 time coordinate, and x^k is a Φ_0 coordinate of a particle at t, then $V^k = \mathrm{d}x^k/\mathrm{d}t$ is the k component of the Φ_0 velocity of the particle at t.

Physical quantities measured in natural units are assumed to be independent of the choice of Φ_0 chart. Of course, it is often convenient to define such quantities in terms of a Φ_0 chart, but we must then make sure that the definition is invariant under change of chart. A quantity Q that is measured in Φ_0 units will in general depend on the choice of Φ_0. If the dimensions of Q involve only mass, length and time—say $[Q] = [\mathrm{L}^{\alpha}\mathrm{T}^{\beta}\mathrm{M}^{\delta}]$, α, β, δ real—and if Q' is the same quantity measured in Φ_0' units, then

$$Q' = Q \exp\left\{\frac{(\alpha-\beta-3\delta)(\Phi_0-\Phi_0')}{c_{\mathrm{E}}^2}\right\}. \tag{2.2}$$

It follows from (2.2) and the dimensional homogeneity of physical equations that any physical equation which is valid for Φ_0 quantities remains valid when each Φ_0 quantity is replaced by the corresponding Φ_0' quantity.

The natural units at the point p where the potential is Φ_p are the same as the Φ_0 units if $\Phi_p = \Phi_0$. Thus if Q is a Φ_0 quantity that depends on Φ only through the value Φ_p, and Q_{E} is the corresponding natural quantity, then $Q = Q_{\mathrm{E}}$ when $\Phi_p = \Phi_0$. It follows from (2.2) that for other values of Φ_p

$$Q_{\mathrm{E}} = Q \exp\left\{\frac{(\alpha-\beta-3\delta)(\Phi_0-\Phi_p)}{c_{\mathrm{E}}^2}\right\} \tag{2.3}$$

(cf. I, equation (35)). If Q depends only on the values of Φ and its derivatives at p, one may regard (2.3) as a definition of Q_{E}.

We assume, as in I, that the potential Φ is observationally indistinguishable from the potential $\Phi+k$, where k is any constant. This means that the value of any measurable physical quantity Q_{E} (measured in *natural* units) must be independent of the choice of k. As an example of a measurable quantity we may take $Q_{\mathrm{E}} = \Phi_p - \Phi_q$, the potential difference between the points p and q; but we cannot take $Q_{\mathrm{E}} = \Phi_p$.

Equations that hold in a potential Φ often hold in a potential $\Phi+k$. To make this precise, let Q_{E} be a measurable natural quantity in the potential Φ and let $\bar{Q}_{\mathrm{E}}$ be the corresponding natural quantity in the potential $\Phi+k$. Let us define Q to be the Φ_0 quantity corresponding to Q_{E} and $\bar{Q}$ to be the Φ_0+k quantity corresponding to $\bar{Q}_{\mathrm{E}}$. Now if we suppose that Q and Q_{E} satisfy (2.3) and that $\bar{Q}$ and $\bar{Q}_{\mathrm{E}}$ satisfy the same equation, then since $Q_{\mathrm{E}} = \bar{Q}_{\mathrm{E}}$ by the assumption of the last paragraph, we have $Q = \bar{Q}$. Thus any equation valid in the potential Φ for Φ_0 quantities that satisfy (2.3) is valid in the potential $\Phi+k$ for the corresponding Φ_0+k quantities. From (2.2), the equation is also valid for Φ_0 quantities in the potential $\Phi+k$.

The fact that Q_{E} is the same in the potential Φ as in the potential $\Phi+k$ does not mean that it is independent of Φ. It can be any function of the derivatives of Φ and of the potential differences $\Phi_p - \Phi_q$ for any points p and q. However, if we assume that Q_{E} depends on the potential only through its value Φ_p at the point p, or if we assume that the potential is constant in a certain region and Q_{E} depends only on the value of Φ in that region, then it does follow that Q_{E} is independent of Φ.

An example of a quantity Q_{E} that may depend on Φ is the density of a fluid at the space-time point p. If the fluid at p was at rest at the point q, then the density at p will usually be a function of $\Phi_p - \Phi_q$. Another example, which will be important later, is the natural Lagrangian density $\mathscr{L}_{\mathrm{GE}}$ of the gravitational field. This is defined in terms of a Φ_0 chart, and is a function of the first derivatives of Φ. It may also depend on $\Phi - \Phi_1$, where Φ_1 is some special value of Φ (perhaps the value at 'spatial infinity', or an average value of Φ over all space).

For simplicity, we shall usually assume the existence of a Φ_0 chart (U, χ) for which U is the whole of space–time and $\chi(U) = R^4$. (One can formulate the theory without this assumption.) We use the convention that all quantities except Φ are measured in Φ_0 units unless indicated otherwise (e.g. by a suffix E).

3. The action

The easiest way to develop a consistent field theory is to use an action principle. This is especially true for the systems that we shall deal with, whose action can be expressed in terms of a local Lagrangian density. In this section we summarize the essential results for such systems, emphasizing only the points that differ from conventional, special-relativistic field theory.

Let (U, χ) be a chart†, not necessarily Newtonian, whose coordinates are x^μ, and in which the metric has components $g_{\mu\nu}$. The components of the fields with respect to (U, χ) are q_M $(M = 1, 2, \ldots, N)$, the partial derivative of q_M with respect to its μ argument is $q_{M,\mu}$, and we write $q = (q_1, q_2, \ldots, q_N)$, $Dq = (q_{1,0}, q_{1,1}, \ldots, q_{N,3})$. It is assumed that the action $A(\mathscr{R})$ of the system on an arbitrary region $\mathscr{R} \subset U$ can be written

$$A(\mathscr{R}) = \int_{\chi(\mathscr{R})} c_E^{-1} \mathscr{L}'_E(q(x), Dq(x))\{-\det g(x)\}^{1/2}\, dx \tag{3.1}$$

where $dx = dx^0\, dx^1\, dx^2\, dx^3$, $\det g$ is the determinant of the matrix whose elements are $g_{\mu\nu}$ and $\mathscr{L}_E$ is a function independent of $\mathscr{R}$. (We are excluding any explicit dependence of $\mathscr{L}_E$ on x.) Similarly, if (U', χ') is another chart such that $\mathscr{R} \subset U'$, then there exists a function $\mathscr{L}_E'$ such that

$$A(\mathscr{R}) = \int_{\chi'(\mathscr{R})} c_E^{-1} \mathscr{L}'_E(q'(x'), Dq'(x'))\{-\det g'(x')\}^{1/2}\, dx' \tag{3.2}$$

where the x'^μ are the coordinates of (U', χ'), $q' = (q_1', q_2', \ldots, q_N')$, $Dq' = (q'_{1,0}, q'_{1,1}, \ldots, q'_{N,3})$, etc. Since $\mathscr{R}$ is arbitrary and the Jacobian of the transformation $x \to x'$ is

$$\det\left(\frac{\partial x'^\mu}{\partial x'^\nu}\right) = \left(\frac{\det g}{\det g'}\right)^{1/2}$$

equations (3.1) and (3.2) imply that

$$\mathscr{L}'_E(q'(x'), Dq'(x')) = \mathscr{L}_E(q(x), Dq(x)). \tag{3.3}$$

Equation (3.3) holds for all charts. Now assume that (U, χ) is a Φ_0 chart and (U', χ') is a Φ_0' chart, for some Φ_0, Φ_0'. In general,

$$\left.\begin{aligned} x'^0 &= x^0 \exp\left(\frac{\Phi_0' - \Phi_0}{c_E^2}\right) + a^0 \\ x'^k &= b_{km} x^m \exp\left(-\frac{\Phi_0' - \Phi_0}{c_E^2}\right) + a^k \end{aligned}\right\} \tag{3.4}$$

on $U \cap U'$, where the a^μ, b_{km} are constants and $b_{km} b_{kn} = \delta_{mn}$ (cf. § 2). It follows that

$$\frac{\partial}{\partial x^0} = \exp\left(\frac{\Phi_0' - \Phi_0}{c_E^2}\right) \frac{\partial}{\partial x'^0}$$

and

$$\frac{\partial}{\partial x^m} = b_{km} \exp\left(-\frac{\Phi_0' - \Phi_0}{c_E^2}\right) \frac{\partial}{\partial x'^k}.$$

† We restrict ourselves to charts that belong to the C^∞ differentiable structure. We also assume the existence of all necessary derivatives, the convergence of all integrals, etc.

If therefore we define

$$\frac{\partial_{\rm E}}{\partial x^0} = s^{-1}\frac{\partial}{\partial x^0}, \qquad \frac{\partial_{\rm E}}{\partial x^m} = s\,\frac{\partial}{\partial x^m}$$

$$\frac{\partial_{\rm E}}{\partial x'^0} = s'^{-1}\frac{\partial}{\partial x'^0}, \qquad \frac{\partial_{\rm E}}{\partial x'^m} = s'\,\frac{\partial}{\partial x'^m}$$

where $s = \exp\{(\Phi-\Phi_0)/c_{\rm E}^2\}$ and $s' = \exp\{(\Phi-\Phi_0')/c_{\rm E}^2\}$, we have

$$\frac{\partial_{\rm E}}{\partial x^0} = \frac{\partial_{\rm E}}{\partial x'^0}, \qquad \frac{\partial_{\rm E}}{\partial x^m} = b_{km}\frac{\partial_{\rm E}}{\partial x'^k}:$$

the operation $\partial_{\rm E}/\partial x^\mu$ is independent of Φ_0. It is convenient to write

$$\mathrm{D_E}q = \left(\frac{\partial_{\rm E}q_1}{\partial x^0}, \frac{\partial_{\rm E}q_1}{\partial x^1}, \quad \ldots, \quad \frac{\partial_{\rm E}q_N}{\partial x^3}\right)$$

$$\mathrm{D_E'}q' = \left(\frac{\partial_{\rm E}q_1'}{\partial x'^0}, \frac{\partial_{\rm E}q_1'}{\partial x'^1}, \quad \ldots, \quad \frac{\partial_{\rm E}q_N'}{\partial x'^3}\right)$$

and to define $\hat{\mathscr{L}}_{\rm E}$, for *Newtonian* charts, by

$$\hat{\mathscr{L}}_{\rm E}(q(x), \mathrm{D_E}q(x)) = \mathscr{L}_{\rm E}(q(x), \mathrm{D}q(x)).$$

Equation (3.3) then becomes

$$\hat{\mathscr{L}}_{\rm E}'(q'(x'), \mathrm{D_E'}q'(x')) = \hat{\mathscr{L}}_{\rm E}(q(x), \mathrm{D_E}q(x)).$$

We *assume* that $\hat{\mathscr{L}}_{\rm E}' = \hat{\mathscr{L}}_{\rm E}$. In other words, we assume that the function $\hat{\mathscr{L}}_{\rm E}$ is unique—the same for all Newtonian charts.

$\hat{\mathscr{L}}_{\rm E}$ has the dimensions of energy density, $[\hat{\mathscr{L}}_{\rm E}] = [\mathrm{ML^{-1}T^{-2}}]$. It is therefore consistent with (2.3) to define $\hat{\mathscr{L}}$ by

$$\hat{\mathscr{L}}(q, \mathrm{D_E}q) = s^{-2}\hat{\mathscr{L}}_{\rm E}(q, \mathrm{D_E}q) \tag{3.5}$$

where $s = \exp\{(\Phi-\Phi_0)/c_{\rm E}^2\}$. (We are, of course, including Φ among the fields q: we may take $\Phi = q_1$, for example.) Similarly, we define $\mathscr{L}(q, \mathrm{D}q) = s^{-2}\mathscr{L}_{\rm E}(q, \mathrm{D}q)$. Since $\det g = -s^{-4}$, from (2.1), equation (3.1) can be rewritten as

$$\mathrm{A}(\mathscr{R}) = \int_{\chi(\mathscr{R})} c_{\rm E}^{-1}\hat{\mathscr{L}}(q(x), \mathrm{D_E}q(x))\,\mathrm{d}x = \int_{\chi(\mathscr{R})} c_{\rm E}^{-1}\mathscr{L}(q(x), \mathrm{D}q(x))\,\mathrm{d}x \tag{3.6}$$

for any Newtonian chart (U, χ). We take note that one can regard $\mathrm{d}x/c_{\rm E} = \mathrm{d}t\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3$ as a space–time volume element measured in Φ_0 units. All quantities on the right-hand side of (3.6) are then measured in Φ_0 units—which is consistent because $\mathrm{A}(\mathscr{R})$, with dimensions $[\mathrm{ML^2T^{-1}}]$, has the same value in Φ_0 as in natural units.

To formulate an action principle, we consider a family of transformations that depend smoothly on a real parameter ϵ:

$$\left.\begin{aligned} x^* &= F(x, q(x), \mathrm{D}q(x); \epsilon) \\ q_M^*(x^*) &= H_M(x, q(x), \mathrm{D}q(x); \epsilon)\end{aligned}\right\} \tag{3.7}$$

where $x^* = (x^{*0}, x^{*1}, x^{*2}, x^{*3})$, $F = (F^0, F^1, F^2, F^3)$ and $M = 1, 2, \ldots, N$. We define x^μ and $x^{*\mu}$ to be coordinates in the same Φ_0 chart (U, χ) (that is, the first of equations (3.7) represents a mapping $p \to p^*$, where p and p^* are space–time points such that $\chi(p) = x$, $\chi(p^*) = x^*$). We assume that (3.7) holds for each ϵ in some interval that contains zero, and that

$$\left.\begin{aligned} x^* &= x+\epsilon f(x, q(x), \mathrm{D}q(x)) + \mathrm{O}(\epsilon^2) \\ q^*(x^*) &= q(x)+\epsilon h(x, q(x), \mathrm{D}q(x)) + \mathrm{O}(\epsilon^2)\end{aligned}\right\} \tag{3.8}$$

as $\epsilon \to 0$, where

$$f = (f^0, f^1, f^2, f^3), \qquad q^* = (q_1^*, q_2^*, \ldots, q_N^*), \qquad h = (h_1, h_2, \ldots, h_N).$$

The variation of the functional $\mathrm{A}(\mathscr{R})$ corresponding to the transformations (3.8) is (Gelfand and Fomin 1963, § 37.4)

$$\begin{aligned}\delta\mathrm{A}(\mathscr{R}) = \epsilon c_{\mathrm{E}}^{-1} \int_{\chi(\mathscr{R})} \Bigg(\sum_{M=1}^{N} \Bigg[\frac{\partial \mathscr{L}}{\partial q_M}(q(x), \mathrm{D}q(x)) \\ - \frac{\partial}{\partial x^\mu}\left\{ \frac{\partial \mathscr{L}}{\partial q_{M,\mu}}(q(x), \mathrm{D}q(x)) \right\} \Bigg] \bar{h}_M(x, q(x), \mathrm{D}q(x)) \\ + \frac{\partial}{\partial x^\mu}\Bigg\{ \mathscr{L}(q(x), \mathrm{D}q(x)) f^\mu(x, q(x), \mathrm{D}q(x)) \\ + \sum_{M=1}^{N} \frac{\partial \mathscr{L}}{\partial q_{M,\mu}}(q(x), \mathrm{D}q(x)) \bar{h}_M(x, q(x), \mathrm{D}q(x)) \Bigg\} \Bigg) \, \mathrm{d}x \end{aligned} \tag{3.9}$$

where $\bar{h}_M = h_M - q_{M,\mu} f^\mu$. (One can also write $\delta A(\mathscr{R})$ in terms of $\hat{\mathscr{L}}$, where $\hat{\mathscr{L}}(q, \mathrm{D}_\mathrm{E} q) = \mathscr{L}(q, \mathrm{D}q)$, but this is less convenient.) We assume as our *principle of stationary action* that $\delta\mathrm{A}(\mathscr{R})$ vanishes for any region $\mathscr{R}$ and for any admissible† functions f^μ and h_M that vanish on the boundary of $\mathscr{R}$.

If f^μ and h_M vanish on the boundary of $\mathscr{R}$, the second pair of terms in (3.9) contributes nothing to $\delta\mathrm{A}(\mathscr{R})$, and one derives the field equations

$$\frac{\partial \mathscr{L}}{\partial q_M}(q(x), \mathrm{D}q(x)) - \frac{\partial}{\partial x^\mu}\left\{ \frac{\partial \mathscr{L}}{\partial q_{M,\mu}}(q(x), \mathrm{D}q(x)) \right\} = 0. \tag{3.10}$$

Suppose that $\mathscr{R} \to \mathscr{R}^*$ under (3.8), and write

$$\mathrm{A}^*(\mathscr{R}^*) = \int_{\chi(\mathscr{R}^*)} c_\mathrm{E}^{-1} \mathscr{L}(q^*(x^*), \mathrm{D}q^*(x^*)) \, \mathrm{d}x^*. \tag{3.11}$$

The action $\mathrm{A}(\mathscr{R})$ is said to be *invariant* under the family of transformations (3.8) if $\mathrm{A}^*(\mathscr{R}^*) - \mathrm{A}(\mathscr{R}) = \mathrm{O}(\epsilon^2)$ as $\epsilon \to 0$. From Noether's theorem (Gelfand and Fomin 1963, § 37.5), if the action $\mathrm{A}(\mathscr{R})$ is invariant under the family of transformations (3.8) for an *arbitrary* region $\mathscr{R}$, then on each extremal surface of A (i.e. for each q such that (3.10) is satisfied), one has

$$\begin{aligned}\frac{\partial}{\partial x^\mu}\Bigg\{ \mathscr{L}(q(x), \mathrm{D}q(x)) f^\mu(x, q(x), \mathrm{D}q(x)) \\ + \sum_{M=1}^{N} \frac{\partial \mathscr{L}}{\partial q_{M,\mu}}(q(x), \mathrm{D}q(x)) \bar{h}_M(x, q(x), \mathrm{D}q(x)) \Bigg\} = 0.\end{aligned} \tag{3.12}$$

The proof is simple: we substitute (3.10) in (3.9) and set $\delta\mathrm{A}(\mathscr{R}) = 0$, and use the arbitrariness of $\mathscr{R}$.

For any real constant k, define $I(k)$ by

$$\begin{aligned}I(k) = \int_{x^0 = k} \Bigg\{ \mathscr{L}(q(x), \mathrm{D}q(x)) f^0(x, q(x), \mathrm{D}q(x)) \\ + \sum_{M=1}^{N} \frac{\partial \mathscr{L}}{\partial q_{M,0}}(q(x), \mathrm{D}q(x)) \bar{h}_M(x, q(x), \mathrm{D}q(x)) \Bigg\} \, \mathrm{d}^3x\end{aligned} \tag{3.13}$$

where $\mathrm{d}^3x = \mathrm{d}x^1\, \mathrm{d}x^2\, \mathrm{d}x^3$, and where the integral is over the whole hypersurface $x^0 = k$. (We are assuming that U is the whole of space–time.) If one integrates the left-hand side

† For the definition of the class of admissible functions, see Gelfand and Fomin (1963).

of (3.12) over the region between the hyperplanes $x^0 = a$ and $x^0 = b$ and then applies the divergence theorem, one finds that $I(a) = I(b)$, showing that I is a conserved quantity.

As a special case of (3.8), take

$$x^{*\mu} = x^\mu + \delta_{\mu\nu}\epsilon, \qquad q^*(x^*) = q(x). \tag{3.14}$$

Since $q^*_{M,\mu}(x^*) = q_{M,\mu}(x)$, one has $\mathscr{L}(q^*(x^*), \mathrm{D}q^*(x^*)) = \mathscr{L}(q(x), \mathrm{D}q(x))$. Let us define a new chart (U, χ') with coordinates x' by the equation $x^{*\mu} = x'^\mu + \epsilon\delta_{\mu\nu}$, where $x^* = \chi(p^*)$, $x' = \chi'(p^*)$. It follows from (3.14) that $x' = x$, and that $\chi'(\mathscr{R}^*) = \chi(\mathscr{R})$. Since the Jacobian of the transformation $x^* \to x'$ is 1, equation (3.11) implies that

$$\mathrm{A}^*(\mathscr{R}^*) = \int_{\chi(\mathscr{R})} c_{\mathrm{E}}^{-1}\mathscr{L}(q(x'), \mathrm{D}q(x'))\,\mathrm{d}x' = \mathrm{A}(\mathscr{R}) \tag{3.15}$$

and $\mathrm{A}(\mathscr{R})$ is invariant under (3.14) for arbitrary $\mathscr{R}$. The conditions of Noether's theorem are therefore satisfied by (3.14).

The transformations (3.14) correspond to

$$f^\mu = \delta_{\mu\nu}, \qquad h = 0, \qquad \bar{h}_M = h_M, \qquad -q_{M,\mu}f^\mu = -q_{M,\nu}.$$

Substituting in (3.12), one finds

$$\frac{\partial}{\partial x^\mu}\left\{\mathscr{L}(q(x), \mathrm{D}q(x))\delta_{\mu\nu} - \sum_{M=1}^{N}\frac{\partial\mathscr{L}}{\partial q_{M,\mu}}(q(x), \mathrm{D}q(x))q_{M,\nu}(x)\right\} = 0. \tag{3.16}$$

From (3.13), the functions P_m and $\mathscr{E}$ defined by

$$\begin{aligned} P_m(k) &= c_{\mathrm{E}}^{-1}\int_{x^0=k}\sum_{M=1}^{N}\frac{\partial\mathscr{L}}{\partial q_{M,0}}(q(x), \mathrm{D}q(x))q_{M,m}(x)\,\mathrm{d}^3x \\ \mathscr{E}(k) &= \int_{x^0=k}\left\{-\mathscr{L}(q(x), \mathrm{D}q(x)) + \sum_{M=1}^{N}\frac{\partial\mathscr{L}}{\partial q_{M,0}}(q(x), \mathrm{D}q(x))q_{M,0}(x)\right\}\mathrm{d}^3x \end{aligned} \tag{3.17}$$

are conserved quantities. The P_m have the dimensions of momenta and transform like Φ_0 momentum under change of Φ_0, while $\mathscr{E}$ has the dimensions of energy and transforms like a Φ_0 energy (from (3.5) and (2.3)). We define the vector $\boldsymbol{P}$, whose components in (U, χ) are P_m, to be the *total* Φ_0 *momentum* of the system and $\mathscr{E}$ to be the *total* Φ_0 *energy*.

The (*mixed*) Φ_0 *energy–momentum* T is defined to be the (1,1) tensor field† whose components in the Φ_0 chart (U, χ) are given by

$$T^\mu_\nu(x) = -\mathscr{L}(q(x), \mathrm{D}q(x))\delta_{\mu\nu} + \sum_{M=1}^{N}\frac{\partial\mathscr{L}}{\partial q_{M,\mu}}(q(x), \mathrm{D}q(x))q_{M,\nu}(x). \tag{3.18}$$

In terms of the T_ν, one can write (3.16) and (3.17) as

$$T^\mu_{\nu,\mu} = 0 \tag{3.19}$$

$$P_m = c_{\mathrm{E}}^{-1}\int_{x^0=k} T^0_m\,\mathrm{d}^3x, \qquad \mathscr{E} = \int_{x^0=k} T^0_0\,\mathrm{d}^3x. \tag{3.20}$$

Just as in special relativity, the energy–momentum is not uniquely determined by (3.19) and (3.20). Given any functions $\psi_{\mu\nu\pi}$ such that $\psi_{\mu\nu\pi} = -\psi_{\pi\nu\mu}$ for all μ, ν, π, one can define a new energy–momentum $\tilde{T}$ by $\tilde{T}^\mu_\nu = T^\mu_\nu + \psi_{\mu\nu\pi,\pi}$. Since $\psi_{\mu\nu\pi,\pi\mu} = 0$ and $\psi_{0\nu0} = 0$, we have $\tilde{T}^\mu_{\nu,\mu} = 0$ and $\tilde{T}^0_\nu = T^0_\nu + \psi_{0\nu m,m}$. Provided that the functions $\psi_{0\nu m}$ vanish fast enough at spatial infinity, it follows from the divergence theorem that $P_m = c_{\mathrm{E}}^{-1}\int\tilde{T}^0_m\,\mathrm{d}^3x$, $\mathscr{E} = \int\tilde{T}^0_0\,\mathrm{d}^3x$.

† An (m, n) tensor is one with contravariant order m and covariant order n.

To find other families of transformations (3.8) under which $A(\mathscr{R})$ is invariant for arbitrary $\mathscr{R}$, one needs a more elaborate argument. Let (U, χ) and (U', χ') be Φ_0 charts such that $\mathscr{R} \subset U$ and $\chi(\mathscr{R}) \subset \chi'(U')$. Define a mapping $\mu:\mathscr{R} \to \mathscr{R}^*$ by requiring that, for all $p \in \mathscr{R}$, $\mu(p) = p^*$, where $\chi(p) = \chi'(p^*)$. If one writes $x = \chi(p), x' = \chi'(p), x^* = \chi(p^*), x^{*\prime} = \chi'(p^*)$, then $x = x^{*\prime}$: the (U, χ) coordinates of p are the (U', χ') coordinates of p^*. Before the variation, the field components at p in the chart (U, χ) are $q_M(x)$ and those in the chart (U', χ') are $q'_M(x')$. After the variation, the field components at p^* in (U, χ) are $q^*_M(x^*)$, and those in (U', χ') are $q^{*\prime}_M(x^{*\prime})$. We define the variation of the q_M by requiring that

$$q^{*\prime}_M(x^{*\prime}) = q_M(x). \tag{3.21}$$

(The varied field components at p^* in (U', χ') are the same as the unvaried ones at p in (U, χ).) However, for fixed Φ_0, $\hat{\mathscr{L}}$ is the same for all Φ_0 charts, and in terms of the Φ_0 chart (U', χ') we find that

$$A^*(\mathscr{R}^*) = \int_{\chi'(\mathscr{R}^*)} c_E^{-1} \hat{\mathscr{L}}(q^{*\prime}(x^{*\prime}), D'_E q^{*\prime}(x^{*\prime}) \, dx^{*\prime}. \tag{3.22}$$

Since $x^{*\prime} = x$, equation (3.21) can be written as $q^{*\prime}_M = q_M$, or as $q^{*\prime} = q$; and it follows that $q^{*\prime}_{M,\mu} = q_{M,\mu}$, and $D'_E q^{*\prime} = D_E q$. Substituting these results in (3.22), using $\chi'(\mathscr{R}^*) = \chi(\mathscr{R})$, and changing the integration variable to x, one finds that the action is indeed invariant:

$$A^*(\mathscr{R}^*) = \int_{\chi(\mathscr{R})} c_E^{-1} \hat{\mathscr{L}}(q(x), D_E q(x)) \, dx = A(\mathscr{R}). \tag{3.23}$$

In order to apply these results to the construction of conserved quantities, we consider a family of Φ_0 charts (U', χ'), each labelled by a real parameter ϵ. (To be more precise, we should write the family as $\{(U_\epsilon, \chi_\epsilon)\}$, where $(U_\epsilon, \chi_\epsilon)$ is a Φ_0 chart for each ϵ in some neighbourhood of zero.) The coordinates x' of (U', χ') are related to the coordinates x of a fixed Φ_0 chart (U, χ) by

$$x' = x + \epsilon f(x) + O(\epsilon^2) \tag{3.24}$$

as $\epsilon \to 0$, where f is independent of ϵ. Using the previous notation, we have

$$x^{*\prime} = x^* + \epsilon f(x^*) + O(\epsilon^2)$$

and since $x^{*\prime} = x$, we have $x = x^* + \epsilon f(x^*) + O(\epsilon^2)$, and hence

$$x^* = x - \epsilon f(x) + O(\epsilon^2) \tag{3.25}$$

as $\epsilon \to 0$. (Note that (3.25) reduces to (3.14) if one puts $f^\mu = -\delta_{\mu\nu}$ and neglects the terms in ϵ^2.) The action $A(\mathscr{R})$ is invariant under (3.25), and the conditions of Noether's theorem are satisfied, provided that $\mathscr{R} \subset U$ and $\chi(\mathscr{R}) \subset \chi'(U')$ for each chart (U', χ'). One can therefore construct conserved quantities as in (3.13).

As a special case, we discuss the conservation of angular momentum. It follows from § 2 that if (U, χ) is a Φ_0 chart with coordinates x, and if x' is defined by $x'^0 = x^0, x'^k = b_{km} x^m$, where b_{km} is any constant orthogonal 3×3 matrix, then the chart (U, χ') with coordinates x' is a Φ_0 chart. Since the matrix $(\delta_{km} - \epsilon\epsilon_{knm})$, where n is fixed and ϵ_{knm} is the permutation symbol, is orthogonal to first order in ϵ, a special family of transformations (3.24) is

$$x'^0 = x^0, \qquad x'^k = (\delta_{km} - \epsilon\epsilon_{knm}) x^m. \tag{3.26}$$

The corresponding family of mappings (3.25) is

$$x^{*0} = x^0, \qquad x^{*k} = x^k + \epsilon\epsilon_{knm} x^m + O(\epsilon^2) \tag{3.27}$$

as $\epsilon \to 0$ (rotation through ϵ about the x^n axis). One can therefore put $f^\mu = \delta_{\mu k}\epsilon_{knm} x^m$, $\bar{h}_M = h^{(n)}_M - q_{M,k}\epsilon_{knm} x^m$ in Noether's theorem, and equation (3.12) becomes

$$\frac{\partial}{\partial x^\mu} \left\{ -\epsilon_{knm} T^\mu_k(x) x^m + \sum_{M=1}^{N} \frac{\partial \mathscr{L}}{\partial q_{M,\mu}} (q(x), Dq(x)) h^{(n)}_M(x, q(x), Dq(x)) \right\} = 0 \tag{3.28}$$

where we have used the definition (3.18) of the energy–momentum. The $h_M^{(n)}$ are defined by (3.8), and from (3.21), (3.25), (3.24) one finds

$$\epsilon h_M^{(n)}(x, q(x), \mathrm{D}q(x)) = q_M^*(x^*) - q_M^{*\prime}(x^{*\prime}) + \mathrm{O}(\epsilon^2) = q_M(x) - q_M'(x') + \mathrm{O}(\epsilon^2). \tag{3.29}$$

Thus the $h_M^{(n)}$ may be calculated as soon as one knows how the field components q_M behave under the coordinate transformation (3.24). In the special case when q_M is an invariant, one has $h_M^{(n)} = 0$.

If we again assume that U and U′ are the whole of space–time, it follows from (3.28) and (3.13) that the vector $\boldsymbol{J}$, whose components in the chart (U, χ) at the instant $t = k/c_{\mathrm{E}}$ are

$$J_n(k) = -c_{\mathrm{E}}^{-1} \int_{x^0=k} \left\{ \epsilon_{npm} T_p^0(x) x^m + \sum_{M=1}^{N} \frac{\partial \mathscr{L}}{\partial q_{M,0}} (q(x), \mathrm{D}q(x)) h_M^{(n)}(x, q(x), \mathrm{D}q(x)) \right\} \mathrm{d}^3 x \tag{3.30}$$

is a quantity with the dimensions of angular momentum that transforms as a Φ_0 angular momentum under change of Φ_0. We call $\boldsymbol{J}$ the *total* Φ_0 *angular momentum* of the system.

As mentioned in the paragraph following (3.20), if one defines $\tilde{T}_\nu^\mu = T_\nu^\mu + \psi_{\mu\nu\pi,\pi}$, where $\psi_{\mu\nu\pi} = -\psi_{\pi\nu\mu}$, then $\tilde{T}$ satisfies equations (3.19) and (3.20). It is well known (see Landau and Lifshitz 1962, § 32, or Rzewuski 1958) that one can choose the functions $\psi_{\mu\nu\pi}$ so that (3.30) and (3.28) can be written

$$J_n = -c_{\mathrm{E}}^{-1} \int \epsilon_{npm} \tilde{T}_p^0(x) x^m \, \mathrm{d}^3 x \quad \text{and} \quad \partial_\mu(\epsilon_{npm} \tilde{T}_p^\mu(x) x^m) = 0.$$

Using $\tilde{T}_{k,\mu}^{\mu} = 0$, we then find that $\tilde{T}_k^p = \tilde{T}_p^k$.

The conservation laws and symmetries that we have discussed are generalizations of special-relativistic results. However, the other conservation laws and symmetries of special relativity that are associated with invariance under the full Poincaré group are not usually valid in the present theory.

4. The gravitational field

As a first application of the general theory, we discuss the gravitational field. We consider a system of fields whose Φ_0 Lagrangian density $\mathscr{L}$ can be written as a sum $\mathscr{L} = \mathscr{L}_{\mathrm{G}} + \mathscr{L}_{\mathrm{F}}$, where $\mathscr{L}_{\mathrm{G}}$ depends only on the gravitational potential Φ and its first partial derivatives $\Phi_{,\mu}$, and $\mathscr{L}_{\mathrm{F}}$ contains no term that depends only on Φ or its partial derivatives. (This rather vague characterization of $\mathscr{L}_{\mathrm{F}}$ is sufficient for the present, general discussion; we shall be more precise later, when we discuss particular systems of fields.) We write $\mathscr{L}_{\mathrm{G}}(\Phi, \mathrm{D}\Phi)$ instead of $\mathscr{L}_{\mathrm{G}}(q, \mathrm{D}q)$, and we call $\mathscr{L}_{\mathrm{G}}$ the Φ_0 *Lagrangian density of the gravitational field*. From (3.5), the natural Lagrangian density of the system of fields is $\mathscr{L}_{\mathrm{E}} = s^2\mathscr{L}$. Since $\mathscr{L}_{\mathrm{E}}$ is independent of the choice of Newtonian chart, it follows that the natural Lagrangian densities $\mathscr{L}_{\mathrm{GE}} = s^2\mathscr{L}_{\mathrm{G}}$ and $\mathscr{L}_{\mathrm{FE}} = s^2\mathscr{L}_{\mathrm{F}}$ are also independent of this choice.

To determine $\mathscr{L}_{\mathrm{GE}}$, one must make further assumptions. We recall that, in special relativity, if $g^{\mu\nu}\Psi_{,\mu}\Psi_{,\nu}$ is the Lagrangian density of a scalar field Ψ, then Ψ satisfies the wave equation. If, therefore, we suppose that the gravitational potential Φ satisfies the wave equation in the limiting case when the gravitational field is everywhere weak and no other fields are present, then it may be reasonable to assume that $\mathscr{L}_{\mathrm{GE}}$ is a function of $g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}$. To be more precise, the components $g^{\mu\nu}$ of the contravariant metric tensor are determined in a Φ_0 chart (U, χ) by (2.1) and the equations $g^{\mu\pi}g_{\nu\pi} = \delta_{\mu\nu}$:

$$g^{mn} = s^2\delta_{mn}, \qquad g^{\mu 0} = g^{0\mu} = -s^{-2}\delta_{\mu 0}. \tag{4.1}$$

We define a function ω by

$$\omega = g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} \tag{4.2}$$

where $\Phi_{,\mu}$ is the partial derivative with respect to the coordinate x^μ of (U, χ), and we assume that $\mathscr{L}_{GE} = f(\omega, \Phi - \Phi_1)$, where f is a smooth function and Φ_1 is a constant independent of the choice of Newtonian chart. Since ω is independent of the choice of Newtonian chart, so too is $\mathscr{L}_{GE}$.

In this section we are concerned mainly with the gravitational field. Later we shall discuss various non-gravitational fields, but for the present we restrict ourselves to a very simple one: an almost stationary mass distribution. By *almost stationary* we mean that the kinetic energy of the mass distribution is negligible, so that its Lagrangian density is $\mathscr{L}_F = -\epsilon$ where ϵ is its Φ_0 energy density when it is at rest with respect to a Φ_0 chart. Its natural energy density is defined by $\epsilon_E = s^{-2}\epsilon$. It follows from our general assumptions that ϵ_E is independent of the choice of Φ_0, but it is not independent of Φ (see the discussion near the end of § 2). To determine the Φ dependence of ϵ_E, one has to impose an additional condition on the mass distribution. Recalling that in I we developed particle mechanics on the assumption that the natural proper mass of a particle is constant in time, we assume that ρ^*, the Φ_0 density of natural mass of the mass distribution, is constant in time. The Φ_0 mass density ρ is the Φ_0 mass per unit Φ_0 volume of 3-space, and since ρ^* is the natural mass per unit Φ_0 volume of 3-space, one has $\rho = \rho^* s^{-3}$ from (2.3). For a stationary body, assume that ρ is related to the Φ_0 energy density ϵ by $\epsilon = \rho c^2$ (the analogue of the Einstein relation in special relativity). Using $c = s^2 c_E$, we find that $\epsilon = s c_E{}^2 \rho^*$ and

$$\epsilon(x) = P(x) \exp\left\{\frac{(\Phi(x) - \Phi_1)}{c_E^2}\right\}$$

where P is a function independent of Φ and Φ_1 is a constant.

The preceding argument may seem slipshod. If the reader prefers, he may regard the last equation as a limiting case of the properly derived results of § 6 (see (6.23)).

The simplest choice of $\mathscr{L}_G$ is $\mathscr{L}_G = K\omega s^{-2}$, where K is a constant. If one takes $\mathscr{L}_F = -\epsilon$, with ϵ defined as above, the total Lagrangian density for the system is $\mathscr{L} = K\omega s^{-2} - \epsilon$. From (3.10), the field equation for Φ is

$$\Phi_{,mm} - s^{-4}(\Phi_{,00} - 2c_E^{-2}\Phi_{,0}^2) = -\frac{\epsilon}{2Kc_E^2}. \tag{4.3}$$

Since (4.3) must reduce to Poisson's equation when the gravitational field is weak and time-independent, we have $K = -1/8\pi G_E$, where G_E is the Newtonian gravitational constant measured in natural units. With this value of K, the energy–momentum T_G of the gravitational field is given by (3.18):

$$T^{\nu}_{G\mu} = Ks^{-2}(2g^{\pi\nu}\Phi_{,\pi}\Phi_{,\mu} - \delta_{\mu\nu}g^{\pi\lambda}\Phi_{,\pi}\Phi_{,\lambda}). \tag{4.4}$$

The energy density of the gravitational field is $\epsilon_G = T^0_{G0}$. It is positive definite.

If Φ is time-independent and spherically symmetric about the origin, and $\epsilon(x) = 0$ when $r = (x^k x^k)^{1/2} > r_0$, the solution of (4.3) regular for $r > r_0$ is $\Phi(x) = \Phi_\infty - l/r$, where Φ_∞ and l are constants. As pointed out in I, the metric corresponding to such a potential is experimentally indistinguishable from the Schwarzschild metric of the Einstein theory. (It predicts the same perihelion advance of planets, the same bending of light, etc.).

In the field equation (4.3), the energy density of the gravitational field does not appear. There is no very compelling reason why it should, but if one feels on moral grounds that all forms of energy ought to act as sources of the gravitational field, then one will have to change $\mathscr{L}_G$. The most obvious thing to try is $\mathscr{L}_{GE} = F(\omega)$ for some suitable function F, but this does not seem to work. We therefore stick to our previous assumption that $\mathscr{L}_{GE} = f(\omega, \Phi - \Phi_1)$, where Φ_1 is a constant independent of the choice of Newtonian chart. Choosing a simple f, we assume that

$$\left.\begin{aligned} \mathscr{L} &= \mathscr{L}_G + \mathscr{L}_F, \qquad \mathscr{L}_F = -\epsilon \\ \mathscr{L}_G &= K\omega \exp[c_E^{-2}\{-2(\Phi - \Phi_0) + \alpha(\Phi - \Phi_1)\}] \end{aligned}\right\} \tag{4.5}$$

where ϵ is defined as before, and α is a constant. An advantage of assuming that $\mathscr{L}_G$ has an exponential dependence on $\Phi-\Phi_1$ is that we do not have to worry about the value of Φ_1. Replacing Φ_1 by Φ_1+j is equivalent to replacing K by $K\exp(-\alpha j)$, for any constant j. It therefore simply changes the ratio of the terms $\mathscr{L}_G$ and $\mathscr{L}_F$, which we determine by comparison with experiment (or by requiring that the theory reduce to the Newtonian theory in the appropriate limit).

The field equation for Φ that follows from (4.5) is

$$\Phi_{,mm}-s^{-4}\{\Phi_{,00}-\tfrac{1}{2}c_E^{-2}(4-\alpha)\Phi_{,0}^{\,2}\} = -(2Kc_E^2)^{-1}[\alpha K\Phi_{,p}\Phi_{,p}+\epsilon\exp\{-\alpha c_E^{-2}(\Phi-\Phi_1)\}]. \quad (4.6)$$

Assuming that Φ is time-independent and that the gravitational field is everywhere weak, we have $\Phi_{,p}\Phi_{,p}\simeq 0$, and $\Phi_{,mm}\simeq -(2Kc_E^2)^{-1}\epsilon\exp\{-\alpha c_E^{-2}(\Phi-\Phi_1)\}$. If also there exists a constant Φ_∞ such that $\Phi(x)\to\Phi_\infty$ as $r=(x^kx^k)^{1/2}\to\infty$, then because of the weakness of the field one has $\Phi(x)\simeq\Phi_\infty$ for all x. Choosing $\Phi_0=\Phi_\infty$, we get $\epsilon\simeq\epsilon_E$, and the field equation becomes

$$\Phi_{,mm}\simeq -(2Kc_E^2)^{-1}\epsilon_E\exp\{-\alpha c_E^{-2}(\Phi_\infty-\Phi_1)\}.$$

This is Poisson's equation provided that

$$1/K = -8\pi G_E\exp\{\alpha c_E^{-2}(\Phi_\infty-\Phi_1)\}. \quad (4.7)$$

It is rather natural to assume that $\Phi_1=\Phi_\infty$, so that $1/K=-8\pi G_E$ just as before.

The energy–momentum of the gravitational field is given by (3.18):

$$\begin{aligned} T_{G\mu}^{\;m} &= K\{2\Phi_{,m}\Phi_{,\mu}-\delta_{m\mu}(\Phi_{,p}\Phi_{,p}-s^{-4}\Phi_{,0}^{\,2})\}\exp\{c_E^{-2}\alpha(\Phi-\Phi_1)\}\\ T_{G\mu}^{\;0} &= K\{-2s^{-4}\Phi_{,0}\Phi_{,\mu}-\delta_{0\mu}(\Phi_{,p}\Phi_{,p}-s^{-4}\Phi_{,0}^{\,2})\}\exp\{c_E^{-2}\alpha(\Phi-\Phi_1)\}. \end{aligned} \quad (4.8)$$

The energy density of the gravitational field is $\epsilon_G=T_{G0}^{\;0}$, and is again positive definite. Using (4.7) (that is, assuming that $\Phi(x)\to\Phi_\infty$ as $r\to\infty$), we find

$$\epsilon_G = (8\pi G_E)^{-1}(\Phi_{,p}\Phi_{,p}+s^{-4}\Phi_{,0}^{\,2})\exp\{c_E^{-2}\alpha(\Phi-\Phi_\infty)\}. \quad (4.9)$$

Equations (4.8), (4.6) and $\epsilon_G=T_{G0}^{\;0}$ imply

$$\Phi_{,mm}-s^{-4}\{\Phi_{,00}-c_E^{-2}(2-\alpha)\Phi_{,0}^{\,2}\} = -(2Kc_E^2)^{-1}(-\alpha\epsilon_G+\epsilon)\exp\{-\alpha c_E^{-2}(\Phi-\Phi_1)\}. \quad (4.10)$$

If one thinks it reasonable that the energy densities ϵ and ϵ_G should behave in the same way as sources of the gravitational field, then one will take $\alpha=-1$. If we assume (4.7), the only arbitrary constants that remain in the field equation are Φ_∞, the value of Φ 'at spatial infinity' and G_E. In I we solved the static, spherically symmetric case of (4.10). It was shown that when $\alpha=-1$ the perihelion advance of test particles is 8% less than that predicted by the Einstein theory. The bending of light is not appreciably different.

We show in § 5 (following equation (5.11)) that electromagnetic energy is twice as effective a source of Φ as is the energy of a stationary mass distribution. It is possible that ϵ_G acts as a source of Φ in the same way as the electromagnetic energy density, which would mean choosing $\alpha=-2$. The perihelion advance of test particles in a static, spherically symmetric gravitational potential is then 16% less than in the Einstein theory. The bending of light is again not affected.

The field equations that we have considered all reduce to the wave equation when the gravitational field is weak and source free. Physically speaking, this means that small gravitational waves travel at the speed of light. There is at present no experimental evidence for this assumption, and one may choose not to make it. An example of an alternative approach is that of Papapetrou (1954). He assumes that the metric is of the form (2.1) and that the Lagrangian density is the same as in the Einstein theory. The resulting field equation for Φ is elliptic, and does not admit wave-like solutions.

5. The electromagnetic field

We have assumed that the Φ_0 Lagrangian density $\mathscr{L}$ of a system of fields can be written in the form $\mathscr{L}=\mathscr{L}_G+\mathscr{L}_F$, where $\mathscr{L}_G$ and $\mathscr{L}_F$ are the Φ_0 Lagrangian densities of the gravitational and non-gravitational fields respectively. In § 4 we described how one might choose $\mathscr{L}_G$; we now turn to the problem of finding $\mathscr{L}_F$.

One usually deals with fields whose Lagrangian density is known in the special-relativistic limit and, for such fields, there is a simple prescription for finding a possible $\mathscr{L}_F$. One takes the special-relativistic Lagrangian density (which involves the field components q_M and their first partial derivatives $q_{M,\mu}$) and writes each $q_{M,0}$ as $c^{-1}\partial q_M/\partial t$. The x^k and $x^0 = c_E t$ are then reinterpreted as Φ_0 coordinates, the q_M as Φ_0 field components and c as the Φ_0 speed of light. The resulting expression is assumed to be the Φ_0 Lagrangian density $\mathscr{L}_F$. Since

$$c = c_E s^2 = c_E \exp\left\{\frac{2(\Phi - \Phi_0)}{c_E^2}\right\}$$

this amounts to saying that one gets $\mathscr{L}_F$ from the special-relativistic Lagrangian density by replacing $q_{M,0}$ by $s^{-2}q_{M,0}$.

This prescription gives a reasonable first guess for $\mathscr{L}_F$: one that has the correct transformation properties and the correct special-relativistic limit. The guess will sometimes be wrong (just as, in the Einstein theory, the principle of equivalence sometimes gives the wrong answer—see Trautman 1965, § 6.2). To put matters right, one can try replacing terms like $s^p q_{M,\mu}$ by $(s^p q_M)_{,\mu}$, or one can introduce new q_M, until one gets at last an $\mathscr{L}_F$ that agrees with experiments.

As a first example, we consider a system of interacting gravitational and electromagnetic fields. In special relativity, the Lagrangian density of the electromagnetic field interacting with a current density j is $\frac{1}{2}(E^2 - B^2) + c^{-1}A_\mu j^\mu$, where the electric field $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$ are related to the electromagnetic potential $A = (A_0, \boldsymbol{A})$ by

$$\boldsymbol{E} = \nabla A_0 - c^{-1}\partial \boldsymbol{A}/\partial t, \qquad \boldsymbol{B} = \operatorname{curl} \boldsymbol{A}.$$

We assume that the j^μ are given functions; we are not going to discuss the dynamics of the current density.

Electromagnetic quantities will be measured in Heaviside (that is, rationalized Gaussian) units†, in which the force between charges $\mathscr{Q}_1$ and $\mathscr{Q}_2$ a distance r apart has a magnitude $|\mathscr{Q}_1\mathscr{Q}_2/4\pi r^2|$. (We are still considering flat space–time.) The dimensions of charge are therefore $[\mathscr{Q}] = [\mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-1}]$, while $[\boldsymbol{E}] = [\boldsymbol{B}] = [\mathscr{Q}\mathrm{L}^{-2}] = [\mathrm{M}^{1/2}\mathrm{L}^{-1/2}\mathrm{T}^{-1}]$, and $[\boldsymbol{A}] = [\mathrm{L}\boldsymbol{B}] = [\mathrm{M}^{1/2}\mathrm{L}^{1/2}\mathrm{T}^{-1}]$. The j^μ are related to the charge density ρ and the 3-velocity $\boldsymbol{V}$ of the charge density by $j^0 = \rho c$, $\boldsymbol{j} = \rho \boldsymbol{V}$, so that $[j^\mu] = [\mathscr{Q}\mathrm{L}^{-2}\mathrm{T}^{-1}] = [\mathrm{M}^{1/2}\mathrm{L}^{-1/2}\mathrm{T}^{-2}]$. We see that all the terms in the Lagrangian density $\frac{1}{2}(E^2 - B^2) + c^{-1}A_\mu j^\mu$ do in fact have the dimensions of energy density.

If one applies the prescription given at the beginning of this section, one finds that the Φ_0 Lagrangian density of the electromagnetic field is $\mathscr{L}_F = \frac{1}{2}(E^2 - B^2) + c^{-1}A_\mu j^\mu$, where all quantities are now measured in Φ_0 units, and where

$$\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}, \qquad \boldsymbol{E} = \nabla A_0 - c^{-1}\partial \boldsymbol{A}/\partial t = \nabla A_0 - s^{-2}\boldsymbol{A}_{,0}.$$

It follows from (2.3) that A_μ and j^μ have the same values in Φ_0 as in natural units: $A_\mu = A_{\mu\mathrm{E}}$, $j^\mu = j_\mathrm{E}$.

The total Lagrangian density of the system is $\mathscr{L} = \mathscr{L}_G + \mathscr{L}_F$ and, since $\mathscr{L}_G$ depends only on Φ, the field equations for the electromagnetic field are determined entirely by $\mathscr{L}_F$. Using (3.10) and the $\mathscr{L}_F$ of the last paragraph one can show, in much the same way as in the Maxwell theory, that the total Φ_0 charge of the system is a conserved quantity. The proof requires that the fields vanish sufficiently fast as $r = (x^k x^k)^{1/2} \to \infty$.

Conservation of the total Φ_0 charge may seem strange, but it is not obviously impossible. One cannot of course conclude from the conservation of the total, macroscopic, Φ_0 charge that the Φ_0 charges of elementary particles must be constant: it might equally well be that the *natural* charges of elementary particles are constant, and that the conservation of the total Φ_0 charge is accomplished by the annihilation or creation of charged elementary

† Choosing to measure electromagnetic quantities in Heaviside units is like choosing to measure lengths in metres (rather than feet, say). When we discuss electromagnetism in general space–time, we shall have to distinguish between Φ_0 Heaviside units and natural Heaviside units, just as we distinguish between Φ_0 metres and natural metres.

particles. In fact, the assumption that the Φ_0 charge of the electron is constant is not tenable. It implies that the Rydberg 'constant', measured in natural units, is a function of Φ, and this does not agree with measurements of the gravitational red shift.

Following the conservative principles of I, we reject the odd idea that particles are created or destroyed in such a way that the total Φ_0 charge is conserved. Instead, we try to modify the Lagrangian density $\mathscr{L}_{\mathrm{F}}$ so that it implies the conservation of the total *natural* charge. The modification is chosen so that the equation expressing this conservation law shall have as simple a form as possible. In this way we are led to assume that

$$\mathscr{L}_{\mathrm{F}} = \tfrac{1}{2}(E^2 - B^2) + c^{-1} s A_\mu j^\mu \tag{5.1}$$

$$\boldsymbol{E} = s^{-1}\{\nabla(s^2 A_0) - \boldsymbol{A}_{,0}\}, \qquad \boldsymbol{B} = s \operatorname{curl} \boldsymbol{A} \tag{5.2}$$

where now $A_\mu = A_{\mu\mathrm{E}}\, s^{-1}$, and where the j^μ are related to the Φ_0 charge density ρ and its Φ_0 velocity $\boldsymbol{V}$ by

$$j^0 = \rho c, \qquad j^k = \rho V^k. \tag{5.3}$$

To keep the dimensions right in (5.1), one may regard s as a Φ_0 length, for example (one has $s_{\mathrm{E}} = 1$). The dimensions of the electromagnetic potential are then

$$[A_\mu] = [\mathrm{M}^{1/2}\mathrm{L}^{-1/2}\mathrm{T}^{-1}].$$

We assume that the total Lagrangian density is $\mathscr{L} = \mathscr{L}_{\mathrm{F}} + \mathscr{L}_{\mathrm{G}}$, where $\mathscr{L}_{\mathrm{G}}$ is given by (4.5). The electromagnetic field equations that follow from (3.10), (5.1), (5.2) and (5.3) are

$$\operatorname{curl}(s\boldsymbol{B}) = c_{\mathrm{E}}^{-1} s^{-1} \boldsymbol{j} + (s^{-1}\boldsymbol{E})_{,0} \tag{5.4}$$

$$\operatorname{div}(s^{-1}\boldsymbol{E}) = c_{\mathrm{E}}^{-1} s^{-3} j^0 = s^{-1}\rho \tag{5.5}$$

and from (5.2)

$$\operatorname{curl}(s\boldsymbol{E}) = -(s^{-1}\boldsymbol{B})_{,0} \tag{5.6}$$

$$\operatorname{div}(s^{-1}\boldsymbol{B}) = 0. \tag{5.7}$$

We call equations (5.4)–(5.7) the (*generalized*) *Maxwell equations*.

From (5.4) and (5.5) one derives the *continuity equation*

$$\operatorname{div}(s^{-1}\boldsymbol{j}) = -\frac{\partial}{\partial t}(s^{-1}\rho). \tag{5.8}$$

Provided that $s^{-1}\boldsymbol{j}$ falls off sufficiently rapidly at spatial infinity, it follows from (5.8) and the divergence theorem that $\int s^{-1}\rho \, \mathrm{d}^3x$, where the integral is over the whole hyperplane $x^0 = k$, is a conserved quantity (that is, its value is independent of the choice of k). The quantity ρ is the Φ_0 charge per unit Φ_0 volume. Since natural and Φ_0 charge are related by $\mathscr{Q}_{\mathrm{E}} = s^{-1}\mathscr{Q}$, the *natural* charge per unit Φ_0 volume is $s^{-1}\rho$, and the total natural charge at the instant $x^0 = k$ is $\int s^{-1}\rho \, \mathrm{d}^3x$. The total natural charge is therefore conserved.

The energy–momentum T_{F} corresponding to the $\mathscr{L}_{\mathrm{F}}$ of (5.1) is given by

$$\left.\begin{aligned} T_{\mathrm{F}\mu}^{\;\;\nu} &= \frac{\partial \mathscr{L}_{\mathrm{F}}}{\partial A_{\lambda,\nu}} A_{\lambda,\mu} + \frac{\partial \mathscr{L}_{\mathrm{F}}}{\partial \Phi_{,\nu}} \Phi_{,\mu} - \delta_{\mu\nu}\mathscr{L}_{\mathrm{F}}: \\ T_{\mathrm{F}\mu}^{\;\;m} &= sE_m A_{0,\mu} - s\epsilon_{mpn} B_n A_{p,\mu} + 2c_{\mathrm{E}}^{-2} s A_0 E_m \Phi_{,\mu} - \delta_{\mu m}\mathscr{L}_{\mathrm{F}} \\ T_{\mathrm{F}\mu}^{\;\;0} &= -s^{-1} E_p A_{p,\mu} - \delta_{0\mu}\mathscr{L}_{\mathrm{F}} \end{aligned}\right\}. \tag{5.9}$$

From (5.1), (5.2) and (5.9)

$$T_{\mathrm{F}0}^{\;\;0} = \tfrac{1}{2}(E^2 + B^2) - c^{-1} s A_\mu j^\mu - s^{-1} E_m (s^2 A_0)_{,m}. \tag{5.10}$$

The field equation for Φ is $(\partial\mathscr{L}/\partial\Phi_{,\mu})_{,\mu} = \partial\mathscr{L}/\partial\Phi$, where $\mathscr{L} = \mathscr{L}_{\mathrm{G}} + \mathscr{L}_{\mathrm{F}}$, and to calculate $\partial\mathscr{L}_{\mathrm{F}}/\partial\Phi$ one must know how j^μ depends on Φ. Since $\int s^{-1}\rho \, \mathrm{d}^3x$ is a conserved quantity, a possible, very special choice of ρ is $\rho(x) = \exp[c_{\mathrm{E}}^{-2}\{\Phi(x) - \Phi_1\}]R(\boldsymbol{x})$, where Φ_1 is a constant, and R is a function independent of Φ. It follows from (5.3) that

$(\partial/\partial\Phi)(s^{-1}j^0) = 2c_E^{-2}s^{-1}j^0$. Setting $j^k = 0$, and using (5.5) we find that the Φ field equation is (cf. (4.6))

$$\begin{aligned}\Phi_{,mm} - s^{-4}\{\Phi_{,00} - \tfrac{1}{2}(4-\alpha)c_E^{-2}\Phi_{,0}^{\,2}\} + \tfrac{1}{2}\alpha c_E^{-2}\Phi_{,p}\Phi_{,p} \\ = -(2Kc_E^2)^{-1}(E^2+B^2)\exp\{-\alpha c_E^{-2}(\Phi-\Phi_1)\}.\end{aligned} \tag{5.11}$$

Assuming as before that $\Phi \to \Phi_1$, as $r = (x^kx^k)^{1/2} \to \infty$, one has $K = -1/8\pi G_E$ (cf. equations (4.6) and (4.7)). If we identify $\frac{1}{2}(E^2+B^2)$ as the energy density of the electromagnetic field (see equation (5.18)), then (4.6) shows that this is twice as effective a source of Φ as is the energy density ϵ of a matter distribution. (We recall that in I we showed that the weight of a 'photon' is twice that of a slowly moving particle of the same energy.)

Define functions $F_{\mu\nu}$ by

$$\left.\begin{aligned} F_{m0} &= -F_{0m} = (s^2A_0)_{,m} - A_{m,0} = sE_m \\ F_{mn} &= -F_{nm} = A_{n,m} - A_{m,n} = \epsilon_{mnp}s^{-1}B_p \end{aligned}\right\}. \tag{5.12}$$

The Maxwell equations (5.6) and (5.7) are equivalent to

$$F_{\lambda\mu,\nu} + F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} = 0. \tag{5.13}$$

Define functions F^{μ}_{ν} by $F^{\mu}_{\nu} = g^{\mu\lambda}F_{\lambda\nu}$. From (4.1) and (5.12)

$$\left.\begin{aligned} &F^0_0 = 0, \qquad F^m_0 = s^3E_m, \qquad F^0_m = s^{-1}E_m \\ &F^m_n = -F^n_m = \epsilon_{mnp}sB_p \end{aligned}\right\}. \tag{5.14}$$

The Maxwell equations (5.4) and (5.5) are equivalent to

$$\left.\begin{aligned} F^{\mu}_{m,\mu} &= -c_E^{-1}s^{-1}j^m \\ F^0_{m,m} &= (s^{-4}F^{\mu}_0)_{,\mu} = c_E^{-1}s^{-3}j^0 \end{aligned}\right\}. \tag{5.15}$$

Equations (5.13) and (5.15) are sometimes easier to manipulate than (5.4)–(5.7).

The Lagrangian density (5.1) can be written $\mathscr{L}_F = \mathscr{L}_e + \mathscr{L}_{int}$, where $\mathscr{L}_e = \frac{1}{2}(E^2-B^2)$ is the Lagrangian density of the electromagnetic field, and $\mathscr{L}_{int} = c_E^{-1}s^{-1}A_\mu j^\mu$ represents the interaction with the current density. The energy–momentum T_e corresponding to $\mathscr{L}_e$ is defined by $T_{e\mu}^{\ \nu} = (\partial\mathscr{L}_e/\partial A_{\lambda,\nu})A_{\lambda,\mu} + (\partial\mathscr{L}_e/\partial\Phi_{,\nu})\Phi_{,\mu} - \delta_{\mu\nu}\mathscr{L}_e$. We have

$$T_{e\mu}^{\ \nu} = T_{F\mu}^{\ \nu} + \delta_{\mu\nu}\mathscr{L}_{int} \tag{5.16}$$

where T_F is given by (5.9).

The energy–momentum $\hat{T}$ of an *isolated* system can be chosen to have the symmetry property $\hat{T}^m_n = \hat{T}^n_m$, which is related to the conservation of angular momentum (see § 4). For a system that interacts with an external current density, the total angular momentum need not be conserved. However, one can still define a symmetrical energy–momentum $\hat{T}_e$ for the electromagnetic field by

$$\left.\begin{aligned} \hat{T}_{en}^{\ m} &= T_{en}^{\ m} - F^{\lambda}_m A_{n,\lambda} \\ \hat{T}_{e0}^{\ m} &= T_{e0}^{\ m} - F^{\lambda}_m (s^2A_0)_{,\lambda} \\ \hat{T}_{em}^{\ 0} &= T_{em}^{\ 0} + s^{-4}F^{\lambda}_0 A_{m,\lambda} \\ \hat{T}_{e0}^{\ 0} &= T_{e0}^{\ 0} + s^{-4}F^{\lambda}_0 (s^2A_0)_{,\lambda} \end{aligned}\right\}. \tag{5.17}$$

From (5.2), (5.14), (5.16) and (5.9)

$$\left.\begin{aligned} \hat{T}_{en}^{\ m} &= E_mE_n + B_mB_n - \tfrac{1}{2}\delta_{mn}(E^2+B^2) \\ \hat{T}_{e0}^{\ m} &= s^2\epsilon_{mpn}E_pB_n \\ \hat{T}_{em}^{\ 0} &= -s^{-2}\epsilon_{mpn}E_pB_n \\ \hat{T}_{e0}^{\ 0} &= \tfrac{1}{2}(E^2+B^2) \end{aligned}\right\} \tag{5.18}$$

or alternatively, using (5.14),

$$\hat{T}_{\mathrm{e}\nu}^{\ \mu} = s^{-2}(F^{\mu}_{\lambda}F^{\lambda}_{\nu} - \tfrac{1}{4}\delta_{\mu\nu}F^{\pi}_{\lambda}F^{\lambda}_{\pi}). \tag{5.19}$$

It follows that $\hat{T}_{\mathrm{e}m}^{\ n} = \hat{T}_{\mathrm{e}n}^{\ m}$. Indeed, if one defines $\hat{T}_{\mathrm{e}\mu\nu} = g_{\mu\pi}\hat{T}_{\mathrm{e}\nu}^{\ \pi}$, equation (2.1) implies that $\hat{T}_{\mathrm{e}\mu\nu} = \hat{T}_{\mathrm{e}\nu\mu}$.

From (5.19), (5.15), (5.13) we get, after a short calculation,

$$\hat{T}_{\mathrm{e}\nu,\mu}^{\ \mu} = -c_{\mathrm{E}}^{-1}(s^{-1}j^{m}F_{m\nu} + s^{-3}j^{0}F_{0\nu}) - \tfrac{1}{2}F_{p0}F_{p0}(s^{-2})_{,\nu} + \tfrac{1}{4}F_{pl}F_{pl}(s^{2})_{,\nu} \tag{5.20}$$

which is equivalent to

$$\hat{T}_{\mathrm{e}n,\mu}^{\ \mu} = c^{-1}(\epsilon_{nmp}j^{m}B_{p} + j^{0}E_{n}) + c_{\mathrm{E}}^{-2}(E^{2}+B^{2})\Phi_{,n} \tag{5.21}$$

$$\hat{T}_{\mathrm{e}0,\mu}^{\ \mu} = -c_{\mathrm{E}}^{-1}j^{m}E_{m} + c_{\mathrm{E}}^{-2}(E^{2}+B^{2})\Phi_{,0}. \tag{5.22}$$

One can interpret the right-hand side of (5.21) in terms of the force densities of the electromagnetic and gravitational fields, and the right-hand side of (5.22) in terms of the rate at which these force densities do work.

For given $\boldsymbol{E}$, $\boldsymbol{B}$ and Φ, one can regard (5.2) as a set of partial differential equations for the A_{μ}. These equations do not determine the A_{μ} uniquely: there exist functions G_{μ}, not all zero, such that the transformation $A_{\mu} \rightarrow A_{\mu}+G_{\mu}$, $\Phi \rightarrow \Phi$, leaves $\boldsymbol{E}$ and $\boldsymbol{B}$ invariant. Any such transformation is called a *gauge transformation*. The field equations (5.4)–(5.7) and (5.11) are invariant under gauge transformations. As in classical electromagnetism, we may restrict the gauge transformations by imposing a condition on the A_{μ}. If the restriction is such that the G_{μ} are uniquely determined, well and good. Otherwise we must make sure that all observable consequences of the theory are independent of the choice of the G_{μ}.

The necessary and sufficient conditions that must be satisfied by the G_{μ} if the transformation $A_{\mu} \rightarrow A_{\mu}+G_{\mu}$, $\Phi \rightarrow \Phi$, is to be a gauge transformation are

$$(s^{2}G_{0})_{,m} = G_{m,0}, \qquad G_{m,n} = G_{n,m}. \tag{5.23}$$

The Poincaré lemma implies that there exists a function Γ such that

$$G_{m} = \Gamma_{,m}, \qquad G_{0} = s^{-2}\Gamma_{,0}. \tag{5.24}$$

The simplest way of restricting the gauge transformations is perhaps by requiring that the A_{m} satisfy $A_{m,m} = 0$, or $\operatorname{div}\boldsymbol{A} = 0$ (*Coulomb gauge*). The existence of such A_{m} is proved as in classical electromagnetism. If we define $A'_{\mu} = A_{\mu}+G_{\mu}$, where $A'_{m,m} = 0$ and where we impose the same boundary conditions on the A'_{μ} as on the A_{μ} at spatial infinity, then $G_{m,m} = \Gamma_{,mm} = 0$, and $\Gamma_{,\mu}(x) \rightarrow 0$ as $|\boldsymbol{x}| \rightarrow \infty$. Hence $\Gamma_{,\mu mm} = 0$, and it follows from a theorem of analysis that $G_{\mu}(x) = \Gamma_{,\mu}(x) = 0$ for all x. We have therefore proved that $A'_{\mu} = A_{\mu}$: the electromagnetic potential is uniquely determined by these conditions.

6. The ideal fluid

As a final example, we consider the interaction of the gravitational field with an ideal fluid. We use the same Lagrangian method as before, even though this is not the approach favoured by most fluid dynamicists. (For an attack on the use of variational principles, see Truesdell and Toupin 1960, § 231; for a more favourable assessment, Serrin 1959, §§ 14, 15; for a brief history of relativistic fluid mechanics, Schmid 1967). The chief limitation of Lagrangian methods is that they cannot be applied to dissipative systems. One is therefore restricted to fluids without viscosity or heat conductivity.

The most convenient Lagrangian formulation of fluid mechanics is in terms of the Clebsch potential (Clebsch 1859). This was first applied to classical hydrodynamics by Bateman (1932) (see also Itô 1953). The special relativistic generalization is due to Wei (1959) and Tam (1966). Tam's special-relativistic Lagrangian density for an ideal fluid may be written

$$\left.\begin{aligned}(\mathscr{L}_{\mathrm{f}})_{\mathrm{spec}} = & -\tfrac{1}{2}\rho J\gamma^2\left(1-\frac{V^2}{c^2}\right)+\tfrac{1}{2}\rho J-\rho U(\rho,\mathscr{S}) \\ & +\rho c^{-1}\gamma V^k(\phi_{,k}+\mathscr{S}\psi_{,k}+\lambda\xi_{,k})+\rho\gamma(\phi_{,0}+\mathscr{S}\psi_{,0}+\lambda\xi_{,0})\end{aligned}\right\} \tag{6.1}$$

where ρ is the proper mass per unit proper volume, and J, U and $\mathscr{S}$ are, respectively, the proper enthalpy, internal energy and entropy, all per unit proper mass. The functions ϕ, ψ, ξ are the generalized Clebsch potentials, $\boldsymbol{V}$ is the 3-velocity of the fluid, $V = |\boldsymbol{V}|$, and γ is a function to be determined from the field equations.

It was assumed in I that the natural, proper mass of a particle is a conserved quantity. Similarly, it seems reasonable to assume that the total natural proper mass of the fluid is conserved (cf. § 4). The Φ_0 Lagrangian density that one gets by applying the simple prescription of § 5 to (6.1) is not compatible with this condition. However, a minor change puts things right, and we find the following Φ_0 Lagrangian density for an ideal fluid:

$$\mathscr{L}_{\mathrm{f}} = -\tfrac{1}{2}\rho J\gamma^2\left(1-\frac{V^2}{c^2}\right)+\tfrac{1}{2}\rho J-\rho U(\rho,\mathscr{S},\Phi)+\rho c^{-1}\gamma V^k s^5\theta_k+\rho\gamma s^3\theta_0 \tag{6.2}$$

where $s = \exp\{(\Phi-\Phi_0)/c_{\mathrm{E}}^2\}$ as before, and the θ_μ are defined by

$$\theta_\mu = \phi_{,\mu}+\mathscr{S}\psi_{,\mu}+\lambda\xi_{,\mu}. \tag{6.3}$$

As usual, Φ is measured in natural units, but the other quantities in (6.2) are in Φ_0 units. The total Φ_0 Lagrangian density of the system is $\mathscr{L} = \mathscr{L}_{\mathrm{f}}+\mathscr{L}_{\mathrm{G}}$, where $\mathscr{L}_{\mathrm{G}}$ is defined by (4.5).

The field equations for ϕ and J that follow from (6.2), (6.3) and (3.10) imply that

$$(\rho c^{-1}\gamma s^5 V^k)_{,k}+(\rho\gamma s^3)_{,0} = 0 \tag{6.4}$$

$$\gamma = \pm\frac{1}{(1-V^2/c^2)^{1/2}}. \tag{6.5}$$

We shall always choose the positive sign in (6.5). Since ρ is the Φ_0 proper mass per unit Φ_0 proper volume, we see that $\rho\gamma$ is the Φ_0 proper mass per unit Φ_0 volume and, by (2.3), that $s^3\rho\gamma$ is the natural proper mass per unit Φ_0 volume. Applying the divergence theorem to (6.4) between the hypersurfaces $x^0 = a$ and $x^0 = b$, where a and b are constants, one proves that the total natural proper mass of the fluid is conserved, provided that ρ vanishes fast enough at spatial infinity.

Using (6.4), we find that the field equations for ψ, λ, ξ and $\mathscr{S}$ simplify to

$$\left.\begin{aligned}&\frac{\mathrm{D}\mathscr{S}}{\mathrm{D}t} = 0, \qquad \frac{\mathrm{D}\xi}{\mathrm{D}t} = 0, \qquad \frac{\mathrm{D}\lambda}{\mathrm{D}t} = 0 \\ &\frac{\mathrm{D}\psi}{\mathrm{D}t} = c_{\mathrm{E}}\gamma^{-1}s^{-3}\frac{\partial U}{\partial\mathscr{S}}\end{aligned}\right\} \tag{6.6}$$

where for any differentiable function f we define

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \boldsymbol{V}\cdot\nabla f+\frac{\partial f}{\partial t} = V^k f_{,k}+c_{\mathrm{E}}f_{,0}.$$

Equation (6.5) and the field equations for ρ, V^k, γ give

$$-U(\rho,\mathscr{S},\Phi)-\rho\frac{\partial U}{\partial\rho}(\rho,\mathscr{S},\Phi)+c^{-1}\gamma s^5 V^k\theta_k+\gamma s^3\theta_0 = 0 \tag{6.7}$$

$$J\gamma V^k+cs^5\theta_k = 0 \tag{6.8}$$

$$-J\gamma^{-1}+c^{-1}s^5V^k\theta_k+s^3\theta_0 = 0. \tag{6.9}$$

Because the dimensions of U are $[\mathrm{L}^2\mathrm{T}^{-2}]$, the natural proper internal energy per unit natural proper mass is $U_{\mathrm{E}} = Us^{-4}$. We assume that U_{E} is independent of Φ, so that

$\partial U/\partial \Phi = 4c_E^{-2}U$. The classical relation $\mathrm{d}U = T\,\mathrm{d}\mathscr{S} - p\,\mathrm{d}v$, where $v = 1/\rho$, is therefore modified to

$$\mathrm{d}V = T\,\mathrm{d}\mathscr{S} - p\,\mathrm{d}v + 4c_E^{-2}U\,\mathrm{d}\Phi. \tag{6.10}$$

Since $\mathrm{d}v = -\mathrm{d}\rho/\rho^2$, we have

$$\frac{\partial U}{\partial \mathscr{S}} = T, \qquad \frac{\partial U}{\partial \rho} = \frac{p}{\rho^2}. \tag{6.11}$$

Equations (6.7), (6.9), (6.11) give the usual expression for the enthalpy:

$$J = U + \frac{p}{\rho}. \tag{6.12}$$

Taking the differential of (6.12) and using (6.10), we get $\mathrm{d}J = T\,\mathrm{d}\mathscr{S} + \rho^{-1}\,\mathrm{d}p + 4c_E^{-2}U\,\mathrm{d}\Phi$, from which it follows that

$$J_{,\mu} = T\mathscr{S}_{,\mu} + \rho^{-1}p_{,\mu} + 4c_E^{-2}\left(J - \frac{p}{\rho}\right)\Phi_{,\mu}. \tag{6.13}$$

From (6.6), (6.13), we have

$$\frac{\mathrm{D}J}{\mathrm{D}t} - 4c_E^{-2}J\frac{\mathrm{D}\Phi}{\mathrm{D}t} = \rho^{-1}\left(\frac{\mathrm{D}p}{\mathrm{D}t} - 4c_E^{-2}p\frac{\mathrm{D}\Phi}{\mathrm{D}t}\right).$$

The dimensions of J are the same as those of U, so $J_E = Js^{-4}$ and

$$\frac{\mathrm{D}J_E}{\mathrm{D}t} = \rho^{-1}\frac{\mathrm{D}(ps^{-4})}{\mathrm{D}t}. \tag{6.14}$$

It simplifies some calculations to define $u_k = \gamma V^k/c$, $u_0 = -\gamma$. From (6.5)

$$u_k u_k - u_0^2 = -1. \tag{6.15}$$

Equations (6.8) and (6.9) give

$$Ju_0 = -s^3\theta_0, \qquad Ju_k = -s^5\theta_k. \tag{6.16}$$

It follows from (6.3), (6.6), (6.11), and the equation

$$\frac{\mathrm{D}f}{Dt} = c\gamma^{-1}(u_k f_{,k} - s^{-2}u_0 f_{,0})$$

that

$$u_k(\theta_{\mu,k} - \theta_{k,\mu}) - s^{-2}u_0(\theta_{\mu,0} - \theta_{0,\mu}) = -s^{-5}T\mathscr{S}_{,\mu} \tag{6.17}$$

which is the *generalized vorticity equation.* From (6.15) and (6.16)

$$u_k\theta_{k,\mu} - s^{-2}u_0\theta_{0,\mu} = (s^{-5}J)_{,\mu} + 2c_E^{-2}s^{-5}Ju_0^2\Phi_{,\mu}.$$

Substituting this into (6.17) and using (6.13), we derive the *generalized Euler equations*

$$\begin{aligned}\frac{\mathrm{D}\theta_\mu}{\mathrm{D}t} &= c\gamma^{-1}(u_k\theta_{\mu,k} - s^{-2}u_0\theta_{\mu,0}) \\ &= c_E\gamma^{-1}s^{-3}\{\rho^{-1}p_{,\mu} + c_E^{-2}\Phi_{,\mu}(-J + 2Ju_0^2 + 4p\rho^{-1})\}\end{aligned} \tag{6.18}$$

A special case of an ideal fluid is 'dust', which is defined by the condition $p = 0$ everywhere. From (6.14), J_E is constant in time for each particle of dust. We assume that, at some initial instant, J_E is everywhere constant. It follows that J_E is constant everywhere and at all times. Putting $\mu = m$ in (6.18), and using (6.16), one finds

$$\frac{\mathrm{D}}{\mathrm{D}t}(s^{-3}\gamma V^m) = -\gamma s\left(1 + \frac{V^2}{c^2}\right)\Phi_{,m}. \tag{6.19}$$

The derivative $\mathrm{D}/\mathrm{D}t$ denotes the rate of change of quantities associated with a given dust particle. Thus, from I, equation (42), the equation of motion of a dust particle is identical with that of a particle whose natural proper mass is constant, and which is subject only to gravitational forces. We proved in I that the worldlines of such particles are geodesics of the space–time metric g. The same is therefore true of the worldlines of dust particles. Hence, if we wish, we may eliminate the undefined objects 'test particles' from our theory, and replace them by 'small dust clouds'.

We now return to the general ideal fluid ($p \neq 0$). The components of the Φ_0 energy–momentum T_{f} corresponding to the Lagrangian density (6.2) are

$$T_{\mathrm{f}\mu}^{\ \nu} = \frac{\partial \mathscr{L}_{\mathrm{f}}}{\partial \phi_{,\nu}} \phi_{,\mu} + \frac{\partial \mathscr{L}_{\mathrm{f}}}{\partial \psi_{,\nu}} \psi_{,\mu} + \frac{\partial \mathscr{L}_{\mathrm{f}}}{\partial \xi_{,\nu}} \xi_{,\mu} - \delta_{\mu\nu} \mathscr{L}_{\mathrm{f}} = \frac{\partial \mathscr{L}_{\mathrm{f}}}{\partial \theta_{\nu}} \theta_{\mu} - \delta_{\mu\nu} \mathscr{L}_{\mathrm{f}}.$$

From (6.5), (6.9), (6.12) one shows that $\mathscr{L}_{\mathrm{f}} = p$ and, using (6.16),

$$\left.\begin{aligned} T_{\mathrm{f}m}^{\ k} &= -\rho J u_k u_m - \delta_{km} p, & T_{\mathrm{f}0}^{\ 0} &= \rho J u_0^2 - p \\ T_{\mathrm{f}0}^{\ k} &= -\rho J s^2 u_k u_0, & T_{\mathrm{f}k}^{\ 0} &= \rho J s^{-2} u_k u_0 \end{aligned}\right\}. \tag{6.20}$$

The Φ_0 energy density ϵ_{f} of the ideal fluid is defined by $\epsilon_{\mathrm{f}} = T_{\mathrm{f}0}^{\ 0}$. Since $u_0 = -\gamma$, we have

$$\epsilon_{\mathrm{f}} = \rho J \gamma^2 - p = \gamma^2 \left(\rho U + \frac{pV^2}{c^2}\right). \tag{6.21}$$

The field equation for the gravitational potential is $(\partial \mathscr{L}_{\mathrm{G}}/\partial \Phi_{,\mu})_{,\mu} = \partial \mathscr{L}_{\mathrm{G}}/\partial \Phi + \partial \mathscr{L}_{\mathrm{f}}/\partial \Phi$. From (6.2), (6.9), (6.10), (6.12) we get

$$\frac{\partial \mathscr{L}_{\mathrm{f}}}{\partial \Phi} = -c_{\mathrm{E}}^{-2} \left\{\rho J \gamma^2 \left(1 + \frac{V^2}{c^2}\right) - 4p\right\} \tag{6.22}$$

and the field equation is (cf. (4.6), (5.11))

$$\left.\begin{aligned} &\Phi_{,mm} - s^{-4}\{\Phi_{,00} - \tfrac{1}{2}(4-\alpha)c_{\mathrm{E}}^{-2}\Phi_{,0}^{\,2}\} + \tfrac{1}{2}\alpha c_{\mathrm{E}}^{-2}\Phi_{,p}\Phi_{,p} \\ &\quad = -(2Kc_{\mathrm{E}}^2)^{-1} \left\{\rho J \gamma^2 \left(1 + \frac{V^2}{c^2}\right) - 4p\right\} \exp\{-\alpha c_{\mathrm{E}}^{-2}(\Phi - \Phi_1)\} \end{aligned}\right\}. \tag{6.23}$$

If we again assume that $\Phi \to \Phi_1$ as $r = (x^k x^k)^{1/2} \to \infty$, then $K = -1/8\pi G_{\mathrm{E}}$ (see equation (4.7)). Equation (6.23) reduces to (4.6) when $p = 0$ and $V = 0$ (one has $\epsilon = \epsilon_{\mathrm{f}} = \rho J$). As $p \to 0$ and $V \to c$, we have $\rho J \gamma^2 (1 + V^2/c^2) - 4p \to 2\epsilon_{\mathrm{f}}$ (compare (5.11) and the remarks that follow).

7. Conclusion

The theory has now been developed far enough to make it plausible that one could easily rewrite the whole of classical physics and take account of gravitational effects in the same sort of way. The situation is to be contrasted with that of geometrical theories of gravitation, whose assumptions are so different from those of other physical theories that it is very hard to accommodate them to anything else. As mentioned in I, § 1, this has resulted in an unfortunate separation of gravitation from the rest of physics. With our theory, there seems to be no reason why reconciliation should not be complete.

From the technical point of view, the theory has two agreeable features.

(i) Space–time can be treated as though it were flat. Thus one avoids the complexities of Riemannian geometry.

(ii) The gravitational field is described by a single, real function. Calculations are consequently much simpler than in the Einstein theory, for example, and one may hope to make progress with previously intractable problems (such as the quantization of the gravitational field).

It is too much to expect that the theory will long survive unchanged. We may hope that it will bear the same relation to the future theory of gravitation as does electrostatics to Maxwell's theory of electromagnetism. Perhaps the proposed Stanford gyroscope experiment will tell us whether we must introduce a gravitational vector potential.

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